6.1, Exm 1,2

6.1) p, g → densities,

have to show: $\int p \wedge g = 1 - \frac{1}{2}\int |p-g|$ (*)

$|p-g| = (p \vee g) - (p \wedge g)$

(*) $\iff \int p \wedge g = 1 - \frac{1}{2}\int (p \vee g) - (p \wedge g)$

$\iff \int p \wedge g + \frac{1}{2}\int p \vee g - \frac{1}{2}\int p \wedge g = 1$

$\iff \frac{1}{2}\int (p \wedge g) + (p \vee g) = 1$

$\iff \frac{1}{2}\int (p+g) = 1$

$\iff \int (p+g) = 2$

Condition 6,3.

$$\int (\sqrt{p_\theta} - \sqrt{p_\phi})^2 \leq C^2 \|\theta - \phi\|^2$$

Final last year

Problem 1) $X \sim Bin(n,p)$, $Y \sim Bin(n,g)$

independent, $\hat{p} = \frac{X}{n}$, $\hat{g} = \frac{Y}{n}$,

Proove as $n \to \infty$ $\dfrac{(\hat{p}-\hat{g})-(p-g)}{\sqrt{\frac{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}{n}}} \rightsquigarrow N(0,1)$

SLTN $\dfrac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}} \rightsquigarrow N(0,1)$ (CLT)

$\dfrac{\hat{g}-g}{\sqrt{\frac{g(1-g)}{n}}} \rightsquigarrow N(0,1)$

1) $\hat{p} \xrightarrow{a.s.} p$ LLN

By CMT $\hat{p}(1-\hat{p}) \xrightarrow{a.s.} p(1-p)$

Analogously $\hat{g}(1-\hat{g}) \xrightarrow{a.s.} g(1-g)$

By CMT $\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g}) \xrightarrow{a.s.} p(1-p)+g(1-g)$

$\sqrt{\dfrac{p(1-p)}{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}} \xrightarrow{a.s.} \sqrt{\dfrac{p(1-p)}{p(1-p)+g(1-g)}}$ CMT

$\sqrt{\dfrac{g(1-g)}{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}} \xrightarrow{a.s.} \sqrt{\dfrac{g(1-g)}{p(1-p)+g(1-g)}}$

$\left(\dfrac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}, \dfrac{\hat{g}-g}{\sqrt{\frac{g(1-g)}{n}}}\right) \rightsquigarrow (U,W)$

U, W independent $N(0,1)$

$\left(\dfrac{\hat{p}-p}{\sqrt{\frac{p(1-p)}{n}}}, \dfrac{\hat{g}-g}{\sqrt{\frac{g(1-g)}{n}}}, \sqrt{\dfrac{p(1-p)}{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}}, \sqrt{\dfrac{g(1-g)}{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}}\right)$

$\rightsquigarrow \left(U, W, \sqrt{\dfrac{p(1-p)}{p(1-p)+g(1-g)}}, \sqrt{\dfrac{g(1-g)}{p(1-p)+g(1-g)}}\right)$

By CMT ($f(a,b,c,d) = ac - bd$)

$\dfrac{(\hat{p}-\hat{g})-(p-g)}{\sqrt{\frac{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}{n}}} \rightsquigarrow \sqrt{\dfrac{p(1-p)}{p(1-p)+g(1-g)}}\, U - \sqrt{\dfrac{g(1-g)}{p(1-p)+g(1-g)}}\, W \sim N(0,1)$

$N\left(0, \dfrac{p(1-p)}{p(1-p)+g(1-g)}\right) - N\left(0, \dfrac{g(1-g)}{p(1-p)+g(1-g)}\right)$

2) conf interval for $(p-g)$ of level $1-\alpha$

$$\mathbb{P}\left(-z_{\frac{\alpha}{2}} \le \dfrac{(\hat{p}-\hat{g})-(p-g)}{\sqrt{\frac{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}{n}}} \le z_{\frac{\alpha}{2}}\right) \to 1-\alpha$$

$$\left((\hat{p}-\hat{g}) - z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}{n}},\ (\hat{p}-\hat{g}) + z_{\frac{\alpha}{2}}\sqrt{\frac{\hat{p}(1-\hat{p})+\hat{g}(1-\hat{g})}{n}}\right)$$

$\hat{p}-\hat{g} = \frac{X}{n} - \frac{Y}{n}$

$X = \sum_{i=1}^{n} \tilde{X}_i$ $\quad \tilde{X}_i \sim Ber(p)$

$Y = \sum_{i=1}^{n} \tilde{Y}_i$ $\quad \tilde{Y}_i \sim Ber(g)$

$\hat{p}-\hat{g} = \dfrac{\sum_{i=1}^{n}(\tilde{X}_i - \tilde{Y}_i)}{n}$ $\quad Var(\tilde{X}_i - \tilde{Y}_i) = p(1-p)+g(1-g)$

$E(\tilde{X}_i - \tilde{Y}_i) = p-g$

CLT $\rightarrow$ $\dfrac{(\hat{p}-\hat{g})-(p-g)}{\sqrt{\frac{p(1-p)+g(1-g)}{n}}} \rightsquigarrow N(0,1)$

Exm 2)

$X_1, \dots, X_n$ iid with density

$f_{\theta_0}(x) = \frac{1}{2} e^{-|x-\theta_0|}$, p odd number

$\Psi_n^{(p)}(\theta) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \theta)^p \xrightarrow{\mathbb{P}} E(X_1-\theta)^p =: \Psi^{(p)}(\theta)$

1) Show $\Psi_n^{(p)}(\theta)$ has a unique zero.

$(X_i - \theta)^p \to$ decreasing in θ

$\Psi_n^{(p)}(\theta) \to$ decreasing

$\lim_{\theta \to -\infty} \Psi_n^{(p)}(\theta) = \infty$, $\lim_{\theta \to \infty} \Psi_n^{(p)}(\theta) = -\infty$

2) $\hat{\theta}_n^{(p)} \xrightarrow{\mathbb{P}} \theta_0$

$\Psi_n^{(p)}(\theta) \xrightarrow{\mathbb{P}} \Psi^{(p)}(\theta) = E(X_1 - \theta)^p$

$\Psi^{(p)}(\theta) \to$ decreasing

$E((X_1 - \theta_0)^p) = \int_{-\infty}^{\infty} \frac{1}{2}(x-\theta_0)^p e^{-|x-\theta_0|} dx$

$= \int_{-\infty}^{\infty} \underbrace{\frac{1}{2} y^p e^{-|y|}}_{f} dy = 0$

$(f(y) = -f(-y))$

$(f(y) + f(-y) = 0)$

$\int_{-\infty}^{\infty} f(y) dy = \int_{-\infty}^{0} f(y) dy + \int_{0}^{\infty} f(y) dy$

$= \int_{0}^{\infty} f(-y) dy + \int_{0}^{\infty} f(y) dy$

$= \int_{0}^{\infty} (f(-y) + f(y)) dy = 0$

Lemma 4.9)

$\hat{\theta}_n^{(p)} \xrightarrow{\mathbb{P}} \theta_0$

3) $\sqrt{n}(\hat{\theta}_n^{(p)} - \theta_0) \rightsquigarrow$ some normal

$\Psi_n^{(p)}(\theta) = \frac{1}{n} \sum_{i=1}^{n} (X_i - \theta)^p = \mathbb{P}_n \Psi_\theta(x)$

$\Psi_\theta(x) = (x-\theta)^p$

$\Psi(\theta) = E(X_1 - \theta)^p = \mathbb{P}\Psi_\theta$

Proposition 4.11.

Conditions. $\Psi_\theta(x) \to$ twice cont. differentiable

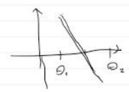
$\dot{\Psi}_\theta(x) = -p(x-\theta)^{p-1}$

$\ddot{\Psi}_\theta(x) = p(p-1)(x-\theta)^{p-2}$

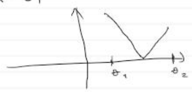
$|\ddot{\Psi}_\theta(x)| \le \ddot{\Psi}(x) \to$ in a neighbourhood of θ_0

θ_1, θ_2, $B = (\theta_1, \theta_2)$, $\theta_1 < \theta_0 < \theta_2$

$|x - \theta|^{p-2}$



$|x-\theta|$



$|x - \theta| \le \max\{|x-\theta_1|, |x-\theta_2|\}$

$|x-\theta|^{p-2} \le \max\{|x-\theta_1|^{p-2}, |x-\theta_2|^{p-2}\}$

$\le |x-\theta_1|^{p-2} + |x-\theta_2|^{p-2}$

Thm 4.19. $\sqrt{n}(\hat{\theta}_n^{(p)} - \theta_0) \rightsquigarrow N\left(0, \frac{\mathbb{P}\Psi_{\theta_0}^2}{(\mathbb{P}\dot{\Psi}_{\theta_0})^2}\right)$

$\mathbb{P}\Psi_{\theta_0}^2 = E(\Psi_{\theta_0}(X))^2 = E(((X-\theta_0)^p)^2) =$

$= E(X-\theta_0)^{2p}$

$= \int_{-\infty}^{\infty} (x-\theta_0)^{2p} \frac{1}{2} e^{-|x-\theta_0|} dx$

$= \int_{-\infty}^{\infty} y^{2p} \frac{1}{2} e^{-|y|} dy$

$= (2p)!$

$E\dot{\Psi}_{\theta_0}(X) = (-p!)$

$AV_p = \frac{(2p)!}{(p!)^2}$

d) $\frac{2p!}{(p!)^2} \cdot \underbrace{\frac{(2p+1)(2p+2)}{(p+1)^2}}_{>1} = \frac{(2p+2)!}{((p+1)!)^2} = \frac{(2(p+1))!}{((p+1)!)^2}$

$p = 1$