Department of Mathematics	retake exam Asymptotic Statistics
VU Amsterdam	February 19, 2020

- **You are not allowed to use calculators, phones, laptops, or other tools.**
- **Clearly put your name and student number on all sheets that you submit.**

1. Consider a parameter $\theta \in \mathbb{R}^k$ and suppose that T_n is an estimator such that

$$\sqrt{n}(T_n - \theta) \rightsquigarrow N_k(0, \Sigma),$$

where Σ is a nonsingular matrix.

(a) Show that $n(T_n - \theta)^T \Sigma^{-1}(T_n - \theta)$ converges in distribution and determine the limiting distribution.

Now suppose that Σ is unknown and that S_n is an estimator for Σ such that $S_n \xrightarrow{P} \Sigma$.

(b) Show that S_n is nonsingular with probability tending to one.

(c) Find an asymptotic $1 - \alpha$ confidence set for θ. Motivate your answer.

2. Let $X_1, \ldots, X_n$ be independent and $N(\theta_0, 1)$-distributed. Define $\hat{\theta}_n$ as the point where the function $\theta \mapsto \sum_{i=1}^n (X_i - \theta)^4$ is minimal.

(a) Show that $\hat{\theta}_n$ is a consistent estimator for θ_0.

(b) Show that $\sqrt{n}(\hat{\theta}_n - \theta_0)$ converges in distribution and determine the limiting distribution.

(c) Determine the asymptotic relative efficiency of $\hat{\theta}_n$ and the sample mean as an estimator for θ_0.

3. (a) Give the precise statement of the general theorem that explains how consistency of M-estimators follows from uniform convergence of the criterion functions that are maximized.

(b) Give a proof of the theorem.

4. Let $X_1, \ldots, X_n$ be independent normal random variables with unknown mean μ and variance 1. Clearly, the distribution function of X_i at the fixed point x is given by $F(x) = \Phi(x - \mu)$, where Φ is distribution function of the standard normal.

We want to compare two estimators for $F(x)$, namely $\mathbb{F}_n(x)$ and $\Phi(x - \bar{X}_n)$, where $\mathbb{F}_n$ is the empirical distribution function.

(a) Intuitively, without a formal computation, which estimator do you expect to be the better of the two? Motivate your answer.

(b) Give the definition of $\mathbb{F}_n(x)$. Show it is a consistent estimator for $F(x)$ and determine its asymptotic variance.

(c) Show that $\Phi(x - \bar{X}_n)$ is a consistent estimator for $F(x)$ and determine its asymptotic variance.

(d) Use your answers of parts (a)–(c) to conjecture an inequality involving the density φ and the distribution function Φ of the standard normal distribution.

5. (a) Give the statement of the James-Stein theorem.

   (b) If $Y \sim N_n(\theta, \sigma^2 I)$, then for the James-Stein estimator $\hat{\theta}_{JS}$ it holds that

$$\mathbb{E}_\theta \|\hat{\theta}_{JS} - \theta\|^2 = \sigma^2 n - \sigma^4 (n-2)^2 \mathbb{E}_\theta \frac{1}{\|Y\|^2}.$$

   (You don't need to prove this statement.) Show that it follows from this expression that

$$\mathbb{E}_\theta \|\hat{\theta}_{JS} - \theta\|^2 \leq 4\sigma^2 + \inf_{c \in \mathbb{R}} \mathbb{E}_\theta \|cY - \theta\|^2.$$

   (Hint: use Jensen's inequality.)

   (c) What is the intuitive meaning of the statement of part (b)?

Norming:

1(a): 3	**2(a): 3**	**3(a): 2**	**4(a): 1**	**5(a): 2**
1(b): 3	**2(b): 3**	**3(b): 4**	**4(b): 4**	**5(b): 3**
1(c): 3	**2(c): 3**		**4(c): 4**	**5(c): 1**
			4(d): 2	