6.2) $p_\theta(x) = f(x-\theta)$,

$$\int \left(\sqrt{p_\lambda(y)} - \sqrt{p_\psi(y)}\right)^2 \le C^2 \|\lambda-\psi\|^2$$

$$\int \left(\sqrt{f(y-\lambda)} - \sqrt{f(y-\psi)}\right)^2 dy$$

$$= \begin{cases} y-\psi = x, & y-\lambda = (y-\psi)+(\psi-\lambda) = x+(\psi-\lambda) \\ \psi-\lambda = \theta \end{cases}$$

$$= \int_{-\infty}^{\infty} \left(\sqrt{f(x+\theta)} - \sqrt{f(x)}\right)^2 dx$$

$$\begin{cases} f>0, \text{ differentiable} \\ \text{then } \left(\sqrt{f(x)}\right)' = \dfrac{f'(x)}{2\sqrt{f(x)}} \end{cases}$$

$$= \int_{-\infty}^{\infty} \left(\int_x^{x+\theta} \frac{f'(t)}{2\sqrt{f(t)}} dt \right)^2 dx$$

$$\left| \int_x^{x+\theta} \frac{f'(t)}{2\sqrt{f(t)}} dt \right|^2 \le \left(\int_x^{x+\theta} 1 \cdot \underbrace{\frac{f'(t)}{2\sqrt{f(t)}}}\ \right)^2 \overset{C\text{-}S}{\le} \int_x^{x+\theta} \frac{f'(t)^2}{4f(t)} dt \int_x^{x+\theta} 1\, dt$$

$$= \theta \int_x^{x+\theta} \frac{f'(t)^2}{4f(t)} dt$$

$$\le \theta \int_{-\infty}^{\infty} \left(\int_x^{x+\theta} \frac{f'(t)^2}{4f(t)} dt \right) dx$$

$$x \in \langle -\infty, \infty \rangle, \quad t \in \langle x, x+\theta \rangle$$

$$t \in \langle -\infty, \infty \rangle, \quad x \in \langle t-\theta, t \rangle$$

$$= \theta \int_{-\infty}^{\infty} dt \int_{t-\theta}^{t} \frac{f'(t)^2}{4f(t)} dx = \theta^2 \underbrace{\int_{-\infty}^{\infty} \frac{f'(t)^2}{4f(t)}}_{C^2}$$

$$\int_{-\infty}^{\infty} \left(\sqrt{f(x-\lambda)} - \sqrt{f(x-\psi)}\right)^2 dx \le C^2 |\lambda-\psi|^2$$

$f>0$, f is differentiable, $\displaystyle\int_{-\infty}^{\infty} \frac{f'(t)^2}{4f(t)} dt < \infty$

6.3, $p_\theta(x) \to$ density of $N(\theta,1)$

$p_\theta(x) = \phi(x-\theta)$, where $\phi = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\phi'(x) = -x \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = -x\phi(x)$$

$$\int_{-\infty}^{\infty} \frac{\phi'(x)^2}{4\phi(x)} dx = \frac{1}{4} \int_{-\infty}^{\infty} \frac{x^2\phi(x)^2}{\phi(x)} = \frac{1}{4} \underbrace{\int_{-\infty}^{\infty} x^2 \phi(x)}_{\text{Var of } N(0,1)} = \frac{1}{4} < \infty$$

6.4) p_θ - density of $Exp(\theta)$

$p_\theta(x) = \theta e^{-\theta x}$, for $x > 0$, θ, λ

$$\int_0^\infty \left(\sqrt{p_\theta(x)} - \sqrt{p_\lambda(x)}\right)^2 dx$$

$$= \int_0^\infty p_\theta(x)\,dx + \int_0^\infty p_\lambda(x)\,dx - 2\int_0^\infty \sqrt{p_\theta(x)}\sqrt{p_\lambda(x)}\,dx$$

$$= 2 - 2\int_0^\infty \sqrt{\theta}\, e^{-\frac{\theta x}{2}} \sqrt{\lambda}\, e^{-\frac{\lambda x}{2}}\,dx$$

$$= 2 - 2\sqrt{\theta}\sqrt{\lambda}\int_0^\infty e^{-\left(\frac{\lambda+\theta}{2}\right)x}\,dx$$

$$= 2 - 2\sqrt{\theta}\sqrt{\lambda}\cdot\frac{2}{\lambda+\theta}$$

$$\left(\frac{1}{2\sqrt{x}}\right)' = \frac{1}{2}\left(x^{-\frac{1}{2}}\right)' = -\frac{1}{4}x^{-\frac{3}{2}}$$

$$= 2 - 4\frac{\sqrt{\theta}\sqrt{\lambda}}{\lambda+\theta}.$$

$0 < \theta_1 < \theta < \theta_2$, neighbourhood (θ_1, θ_2)

$$\sqrt{\theta+\varepsilon} = \sqrt{\theta} + \frac{1}{2\sqrt{\theta}}\cdot\varepsilon + \frac{1}{2}\cdot\frac{1}{\xi^{3/2}}\varepsilon^2$$

$\{\lambda = \theta + \varepsilon\}$

$$= 2 - 4\frac{\theta + \frac{1}{2}\varepsilon}{2\theta+\varepsilon} - \frac{4}{2\theta+\varepsilon}\cdot\frac{1}{2}\frac{1}{\xi^{3/2}}\varepsilon^2$$

$$\leq C\varepsilon^2 = C(\lambda-\theta)^2$$

$2x$

6.5) Adapt thm 6.3.1.

$$\int (\sqrt{p_\theta} - \sqrt{p_\psi})^2 \leq C^2 \|\theta - \psi\|^{2\alpha}, \quad \alpha < 1$$

Start w. interior point θ_0, θ_1 at a distance

For $\hat{\theta}$, $\tilde{\theta}$ defined as θ_0 if $\hat{\theta}$ closer to θ_0.

$$P_\theta(\|\hat{\theta} - \theta\| \geq x) \geq P_\theta(\tilde{\theta} \neq \theta) = E_\theta \, d_{ham}(\theta, \tilde{\theta})$$

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} P(\|\hat{\theta} - \theta\| \geq x) \geq \inf_{\tilde{\theta}} \max_{\theta \in \{\theta_0, \theta_1\}} E_\theta d_{ham}(\tilde{\theta}, \theta)$$

corrolary 6.2.3

$$\inf_{\tilde{\theta}} \max_{\theta \in \{\theta_0, \theta_1\}} E_\theta \, d_{ham}(\tilde{\theta}, \theta) \geq \frac{1}{4} \left(1 - \frac{1}{2} \int (\sqrt{p_{\theta_0}} - \sqrt{p_{\theta_1}})^2\right)^{2n}$$

$$\int (\sqrt{p_{\theta_0}} - \sqrt{p_{\theta_1}})^2 \leq C^2 \|\theta_0 - \theta_1\|^{2\alpha} = C^2 (2x)^{2\alpha}$$
$$= C^2 2^{2\alpha} x^{2\alpha}$$

$$\frac{1}{4}\left(1 - \frac{1}{2}\int (\sqrt{p_{\theta_0}} - \sqrt{p_{\theta_1}})^2\right)^{2n} \geq \frac{1}{4}\left(1 - C' x^{2\alpha}\right)^{2n}$$

$$x^{2\alpha} = \frac{1}{n} \rightarrow x^{2\alpha} = n^{-1} \Rightarrow x = n^{-1/2\alpha}$$

$$\inf_{\hat{\theta}} \sup_{\theta \in \Theta} P_\theta(\|\hat{\theta} - \theta\| \geq n^{-1/2\alpha}) > K$$

6.6) X_i Uniform on $[0,\Theta]$,
minimax rate for estimating θ,

ψ, θ, assume $\psi < \Theta$

$$\int (\sqrt{p_\theta} - \sqrt{p_\psi})^2 \leq \int_{[0,\psi]} \left(\frac{1}{\sqrt{\psi}} - \frac{1}{\sqrt{\theta}}\right)^2 + \int_\psi^\theta \frac{1}{\theta}$$

$$= \left(\frac{1}{\sqrt{\psi}} - \frac{1}{\sqrt{\theta}}\right)^2 \psi + \frac{\theta - \psi}{\theta}$$

$$= \left(\frac{\sqrt{\theta} - \sqrt{\psi}}{\sqrt{\psi\theta}}\right)^2 \cdot \psi + \frac{\theta - \psi}{\theta} \qquad (\sqrt{\theta} - \sqrt{\psi})^2 \leq |\theta - \psi|$$

$$= \frac{|\theta - \psi|}{\psi \cdot \theta} \cdot \psi + \frac{\theta - \psi}{\theta}$$

$$= \frac{2}{\theta} |\theta - \psi| = \frac{2}{\theta} |\theta - \psi|^{2 \cdot \frac{1}{2}}$$

Condition from exercise 6.5, with $\alpha = \frac{1}{2}$

$$n^{-\frac{1}{2\alpha}} = n^{-1}$$

$n(\Theta - X_{(n)}) \rightsquigarrow \text{Exp}$ $\qquad n(X_{(n)} - \Theta) \rightsquigarrow -\text{Exp}$

6.7 Take p with support in $[x_0-1, x_0+1]$ bounded away from 0. Take k bounded, $\int k = 0$, $\int k^2 < \infty$, with support in $[-1,1]$.
Then since p is bounded away from 0, $p + h^m k\left(\frac{x-x_0}{h}\right) > 0$ for h small enough.

$f_0^{(m)}(x_0) = p^{(m)}(x_0)$.

$f_1^{(m)}(x_0) = p^{(m)}(x_0) + k^m(0)$.

Then we just need to make sure $|p^{(m)}(x_0)| < \frac{M}{2}$, $|k^m(0)| < \frac{M}{2}$

b) Adapt proof of 6.4.1, take $d(f,g) = \sqrt{(f(x_0)-g(x_0))^2}$

The submodel is $\{f_0, f_1\}$ which we identify w $\{0,1\}$.

$$\inf_{\hat f} \sup_{f \in \mathcal{F}_{m,M,x_0}} E_f\left(d(f,\hat f)^2\right) \geq \inf_{\hat f} \max_{\theta \in \{0,1\}} E_{f_\theta}\left(d(\hat f, f_\theta)^2\right)$$

$\{d(f_0,f_1)^2 = h^{2m} k(0)^2 = h^{2m} k(0)^2 \, d_{ham}(0,1)\}$

$$\overset{6.2}{\geq} \frac{h^{2m} k(0)^2}{4} \inf_{\hat\theta} \max_{\theta \in \{0,1\}} E_\theta \, d_{ham}(\hat\theta, \theta)$$

$$\overset{6.2.3.}{\geq} \frac{h^{2m} k(0)^2}{4} \cdot \frac{1}{4} \left(1 - \frac{1}{2}\int \left(\sqrt{f_0} - \sqrt{f_1}\right)^2\right)^{2n} \quad (*)$$

By the same argument of 6.4.1.

$$\int \left(\sqrt{f_0} - \sqrt{f_1}\right)^2 \geq C \cdot \int (f_0 - f_1)^2$$
$$= C h^{2m} \int k^2\left(\frac{x-x_0}{h}\right)$$
$$= C h^{2m+1} \int k^2 = C_1 h^{2m+1}$$

$$(*) \geq h^{2m} C_2 \left(1 - C_1 h^{2m+1}\right)^{2n}$$

{ Now to have the constant bounded as n grows need $h^{2m+1} = n^{-1} \Rightarrow h = n^{-\frac{1}{2m+1}}$. }

now using this:

$$\geq n^{-\frac{2m}{2m+1}} C'$$