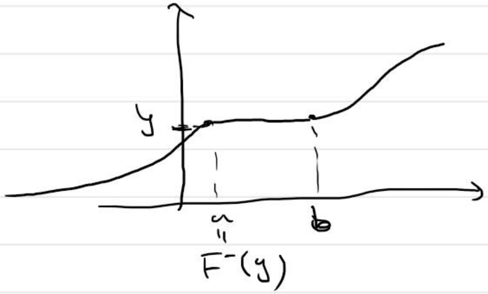
5.1. $\sup_{t \in \mathbb{R}} |\mathbb{F}_n(t) - F(t)|$.

$\mathbb{F}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \le t\}}$

$F^-(u) = \inf\{x \,;\, F(x) \ge u\}$



$F(F^-(y)) = y$.

$x \le F^-(y) \iff F(x) \le F(F^-(y)) = y$

$\sup_{t \in \mathbb{R}} |\mathbb{F}_n(t) - F(t)| = \sup_{y \in \langle 0,1 \rangle} |\mathbb{F}_n(F^-(y)) - F(F^-(y))|$

$= \sup_{y \in \langle 0,1 \rangle} \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{X_i \le F^-(y)\}} - y \right|$

$= \sup_{y \in \langle 0,1 \rangle} \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{F(X_i) \le y\}} - y \right|$ $\quad (F(X_1), \dots, F(X_n))$ $\to$ iid sequence of

$\{$ $F(X_1)$ $\to$ uniform distribution: $\mathbb{P}(F(X_1) \le u)$ $\quad$ $(\{F(X_1) \le u\}$ $= \{X_1 \le F^-(u)\})$

$= \mathbb{P}(X_1 \le F^-(u))$

$F(X_i)$ are iid uniforms $\quad$ $= F(F^-(u)) = u$

$\overset{D}{=} \sup_{y \in \langle 0,1 \rangle} \left| \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_{\{U_i \le y\}} - y \right|$

5.2. $X_1, \dots, X_n$, natural

nonparametric estimate for $\mathbb{P}(X_1 \in B)$

$$\hat{\mathbb{P}}(X_1 \in B) = \underline{\frac{1}{n} \sum \mathbb{1}_{\{X_i \in B\}}} = \underline{\mathbb{P}_n \mathbb{1}_B}$$

$$\mathbb{P}(X_1 \in B) = \underline{\mathbb{P} \mathbb{1}_B} \quad (\quad \mathbb{E}(\mathbb{1}_B(X)))$$

5.3. $X_1,\dots,X_n \sim N(\mu,1)$

$P(X_1\le t) = \Phi(t-\mu)$

1) $F_n(t)$

$\frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i\le t\}}$

$1_{\{X_i\le t\}}$ - Bernoulli $p = P(X_1\le t) = \Phi(t-\mu)$

$Var(1_{\{X_i\le t\}}) = \Phi(t-\mu)(1-\Phi(t-\mu)) = p(1-p)$

$\sqrt{n}(F_n(t)-\Phi(t-\mu)) \underset{CLT}{\rightsquigarrow} N(0,\Phi(t-\mu)(1-\Phi(t-\mu)))$

$\sqrt{n}(\bar{X}_n-\mu)\rightsquigarrow N(0,1)$

$\sqrt{n}(f(\bar{X}_n)-f(\mu))\rightsquigarrow N(0,f'(\mu)^2)$

$\sqrt{n}(\Phi(t-\bar{X}_n)-\Phi(t-\mu))\rightsquigarrow$

$\Phi=\int_{-\infty}^{x}\phi$

$\Phi'(t-x)\cdot(-1)$

2) $\Phi(t-\bar{X}_n)$

$\sqrt{n}(\bar{X}_n-\mu)\rightsquigarrow N(0,1)$

standard normal density

$f(x)=\Phi(t-x)$, $f'(x)=-\phi(t-x)$

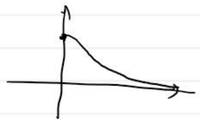
$\sqrt{n}(\Phi(t-\bar{X}_n)-\Phi(t-\mu))\rightsquigarrow N(0,\phi^2(t-\mu))$

Compare the asymptotic variances

$\Phi(t-\mu)(1-\Phi(t-\mu)) \overset{?}{\geq} \phi^2(t-\mu)$

$x=t-\mu$

$\Phi(x)(1-\Phi(x)) \geq \phi^2(x)$ on $[0,\infty)$



$g(x)=\Phi(x)(1-\Phi(x))-\phi^2(x)$

$\lim_{x\to\infty}(\Phi(x)(1-\Phi(x))-\phi^2(x))=0$

$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}} \Rightarrow \phi'(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot(-x)=-x\phi(x)$

$g(x)=\Phi(x)-\Phi^2(x)-\phi^2(x)$

$g'(x)=\phi(x)-2\Phi(x)\phi(x)-2\phi(x)(-x\phi(x))$

$=\phi(x)\underbrace{(1-2\Phi(x)+2x\phi(x))}_{h(x)}$

evaluate at 0 $1-2\cdot\frac{1}{2}+0=0$

$h'(x)=-2\phi(x)+2\phi(x)+2x(-x\phi(x))$

$=-2x^2\phi(x)$

$X_1, \dots, X_n \sim N(\mu, 1)$

$\phi(x - \overline{X}_n)$

$E\left((\phi(x-\overline{X}_n) - \phi(x-\mu))^2 \right)$

$E \int_{\mathbb{R}} (\phi(x-\overline{X}_n) - \phi(x-\mu))^2 \, dx$

$\int_{\mathbb{R}} \phi^2(x-\overline{X}_n)\, dx - 2\int \phi(x-\overline{X}_n)\phi(x-\mu) + \int_{\mathbb{R}} \phi^2(x-\mu) dx$

$\int_{\mathbb{R}} \phi^2(x-\overline{X}_n) dx = \{ z = x - \overline{X}_n \} = \int_{\mathbb{R}} \phi^2(z) dz$

$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{-z^2} dz$

$= \frac{1}{2\pi} \sqrt{2\pi \cdot \tfrac{1}{2}} \underbrace{\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \cdot \frac{1}{2}}} e^{-\frac{z^2}{2 \cdot \frac{1}{2}}}}_{\text{density of } N(0, \frac{1}{2})}$

$= \frac{1}{2\pi} \cdot \sqrt{\pi} = \frac{1}{2\sqrt{\pi}}$

$\int_{\mathbb{R}} \phi^2(x-\mu) = \{ z = x - \mu \} = \int_{\mathbb{R}} \phi^2(z) = \frac{1}{2\sqrt{\pi}}$

$\int_{\mathbb{R}} \phi(x-\overline{X}_n)\phi(x-\mu)\, dx = \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}(x-\overline{X}_n)^2\right) \cdot \exp\left(-\tfrac{1}{2}(x-\mu)^2\right) dx$

$= \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\left((x-\overline{X}_n)^2 + (x-\mu)^2\right)\right) dx$

$= \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\left(2x^2 - 2x(\overline{X}_n + \mu) + \overline{X}_n^2 + \mu^2\right)\right) dx$

$2x^2 - 2x(\overline{X}_n + \mu) = 2(x^2 - x(\overline{X}_n + \mu))$

$= 2\left(x^2 - 2 \cdot x \cdot \frac{\overline{X}_n + \mu}{2}\right)$

$= 2\left(\left(x - \frac{\overline{X}_n + \mu}{2}\right)^2 - \left(\frac{\overline{X}_n + \mu}{2}\right)^2 \right)$

$= \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left(-\tfrac{1}{2}\left(2\left(x - \frac{\overline{X}_n + \mu}{2}\right)^2 - 2\left(\frac{\overline{X}_n + \mu}{2}\right)^2 + \overline{X}_n^2 + \mu^2\right)\right) dx$

$= \frac{1}{2\pi} e^{+\left(\frac{\overline{X}_n + \mu}{2}\right)^2 - \frac{\overline{X}_n^2}{2} - \frac{\mu^2}{2}} \int_{\mathbb{R}} \exp\left(-\left(x - \frac{\overline{X}_n + \mu}{2}\right)^2\right) dx$

$= \frac{1}{2\sqrt{\pi}} e^{-\left(\frac{\overline{X}_n - \mu}{2}\right)^2}$

$MISE = \frac{1}{\sqrt{\pi}} \cdot \left(1 - E e^{-\left(\frac{\overline{X}_n - \mu}{2}\right)^2}\right) \quad (\overline{X}_n \sim N(\mu, \tfrac{1}{n}))$

$\int_{\mathbb{R}} e^{-\left(\frac{z-\mu}{2}\right)^2} \cdot \frac{1}{\sqrt{2\pi \frac{1}{n}}} e^{-\frac{(z-\mu)^2}{2 \cdot \frac{1}{n}}} dz$

$= \sqrt{\frac{2n}{1+2n}}$

$MISE = \frac{1}{\sqrt{\pi}} \left(1 - \sqrt{\frac{2n}{2n+1}}\right)$