Department of Mathematics	midterm exam Asymptotic Statistics
VU	October 23, 2019

- **You are not allowed to use calculators, phones, laptops, or other tools.**
- **Clearly put your name and student number on all sheets that you submit.**

Good luck!

1. Suppose we have random vectors X_n in $\mathbb{R}^k$ such that $X_n \rightsquigarrow N_k(0, I)$.
   (a) Explain what this means by definition.
   (b) Show that in this case $\|X_n\|^2 \rightsquigarrow \chi^2_k$, where $\|\cdot\|$ is the Euclidean norm.

2. Let $X_1, \ldots, X_n$ be i.i.d. with density

$$p_{\lambda,a}(x) = \lambda e^{-\lambda(x-a)} 1_{x \geq a},$$

   where $\lambda > 0$ and $a \in \mathbb{R}$ are unknown parameters.
   (a) Show that the maximum likelihood estimator (MLE) for (a, λ) is given by

$$\left(\hat{a}_n, \hat{\lambda}_n\right) = \left(X_{(1)}, \frac{1}{\overline{X} - X_{(1)}}\right),$$

   where $X_{(1)} = \min\{X_1, \ldots, X_n\}$.
   (b) Show that the MLE is a consistent estimator for the parameter (a, λ).

3. (a) Give the statement of the theorem about the delta-method.
   (b) Give a proof of the theorem.

Continued on the other side...

4. Let $X_1, \ldots, X_n$ be i.i.d., with finite first four moments $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Define the sample variance as

$$S^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

(a) Show that $S^2 = \overline{X^2} - (\overline{X})^2$.

(b) Show that

$$\sqrt{n}\left(\begin{pmatrix}\overline{X}\\ \overline{X^2}\end{pmatrix} - \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\right)$$

converges in distribution and determine the limit.

(c) Determine the weak limit of $\sqrt{n}(S^2 - \sigma^2)$, where $\sigma^2 = \alpha_2 - \alpha_1^2$.

5. Let $X_1, \ldots, X_n$ be an i.i.d. sample from an exponential distribution with parameter $\lambda > 0$. Recall this is the distribution with density $x \mapsto \lambda \exp(-\lambda x)$, mean $1/\lambda$ and variance $1/\lambda^2$.

(a) Show that for $\alpha \in (0, 1)$,

$$\left(\overline{X} - \frac{\overline{X}\xi_\alpha}{\sqrt{n}}, \overline{X} + \frac{\overline{X}\xi_\alpha}{\sqrt{n}}\right)$$

is a confidence interval for $1/\lambda$ of asymptotic level $1 - 2\alpha$. Here ξ_α is the upper-α-quantile of the standard normal distribution.

(b) Give an alternative confidence interval using a variance stabilizing transformation for the sample mean.

(c) What can you say about the relation between the two different intervals?

Norming:

1(a): 2	**2(a): 2**	**3(a): 2**	**4(a): 1**	**5(a): 2**
1(b): 2	**2(b): 4**	**3(b): 3**	**4(b): 2**	**5(b): 2**
			4(c): 4	**5(c): 1**