Asymptotic Statistics

Lecture 5

Recap

Delta method - theorem

Theorem.
Let T_n and T be random vectors in $\mathbb{R}^k$. Suppose that for numbers $r_n \to \infty$, $r_n(T_n - \theta) \rightsquigarrow T$. Suppose that $\varphi : \mathbb{R}^k \to \mathbb{R}^m$ is a (measurable) map that is differentiable at θ, with derivative $\varphi'_\theta : \mathbb{R}^k \to \mathbb{R}^m$. Then

$$r_n(\varphi(T_n) - \varphi(\theta)) - \varphi'_\theta(r_n(T_n - \theta)) \xrightarrow{P} 0$$

and

$$r_n(\varphi(T_n) - \varphi(\theta)) \rightsquigarrow \varphi'_\theta(T).$$

Variance stabilizing transformations

Variance stabilizing transformations

Suppose $\sqrt{n}(T_n - \theta) \overset{P_\theta}{\rightsquigarrow} N(0, \sigma^2(\theta))$. Then can not immediately construct a confidence interval for θ.

If φ is differentiable, then by the delta-method,

$$\sqrt{n}(\varphi(T_n) - \varphi(\theta)) \overset{P_\theta}{\rightsquigarrow} N(0, \varphi'(\theta)^2\sigma^2(\theta)).$$

Idea, find φ such that $\varphi'(\theta)^2\sigma^2(\theta) \equiv 1$, i.e. $\varphi = \int 1/\sigma$. Then can make a confidence interval for $\varphi(\theta)$. Since φ is increasing (why?), transform into confidence interval for θ.

$\rightarrow$ Exercise 3.12

Asymptotic normality of moment estimators

Moment estimators

Have sample $X_1, \ldots, X_n$ from p_θ, $\theta \in \Theta \subset \mathbb{R}^k$. (Generalized) Method of moments: choose functions $f_1, \ldots, f_k$, define $\hat{\theta}_n$ as solution to

$$\begin{array}{ccc} \mathbb{E}_\theta f_1(X_1) & = & \frac{1}{n}\sum_{i=1}^n f_1(X_i) \\ \vdots & \vdots & \vdots \\ \mathbb{E}_\theta f_k(X_1) & = & \frac{1}{n}\sum_{i=1}^n f_k(X_i) \end{array}$$

Usual method of moments: $f_j = x^j$.

Write the system as $e(\theta) = \bar{f}_n$. Then for the solution $\hat{\theta}_n$ to exist, it is necessary that $\bar{f}_n$ is in the range of $e : \Theta \to \mathbb{R}^k$. If e is also injective, then $\hat{\theta}_n = e^{-1}(\bar{f}_n)$.

Moment estimators - asymptotic normality

Idea:

If θ_0 is the true parameter, then by the multivariate CLT, $\sqrt{n}(\bar{f}_n - \mathbb{E}_{\theta_0}\bar{f}_n)$ is asymptotically normal. But $\mathbb{E}_{\theta_0}\bar{f}_n = e(\theta_0)$, hence $\sqrt{n}(\bar{f}_n - e(\theta_0))$ is asymptotically normal.

So if $\hat{\theta}_n = e^{-1}(\bar{f}_n)$ and e^{-1} is differentiable, then by the delta-method,

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = \sqrt{n}(e^{-1}(\bar{f}_n) - e^{-1}(e(\theta_0)))$$

is asymptotically normal as well.

Moment estimators - asymptotic normality

Theorem.
Suppose that $\Theta \subset \mathbb{R}^k$ is open and $e : \Theta \to \mathbb{R}^k$ is injective. If e is continuously differentiable at θ_0 with nonsingular derivative e'_{θ_0} and $\mathbb{E}_{\theta_0} f_j^2(X_1) < \infty$ for all j, then the moment estimator exists with probability tending to 1 and

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \overset{P_\theta}{\rightsquigarrow} N_k(0, e_{\theta_0}^{'-1} \Sigma_{\theta_0} (e_{\theta_0}^{'-1})^T),$$

where Σ_{θ_0} is the covariance matrix of $(f_1(X_1), \ldots, f_k(X_1))$ under P_{θ_0}.

$\to$ Exercise 3.18