Asymptotic Statistics

Lecture 2

Stochastic convergence: so far

Definition.
Let X and X_n be stochastic vectors in $\mathbb{R}^k$.

- X_n converges to X in distribution if $P(X_n \leq x) \to P(X \leq x)$ for all x at which $x \mapsto P(X \leq x)$ is continuous.
  Notation: $X_n \rightsquigarrow X$.
- X_n converges to X in probability if $P(\|X_n - X\| > \varepsilon) \to 0$ for all $\varepsilon > 0$.
  Notation: $X_n \xrightarrow{P} X$.
- X_n converges to X almost surely if $P(X_n \to X) = 1$.
  Notation $X_n \xrightarrow{\text{as}} X$.

Stochastic convergence: so far

Theorem. (Continuous mapping)

Let X and X_n be stochastic vectors in $\mathbb{R}^k$. Let $g : \mathbb{R}^k \to \mathbb{R}^m$ be measurable and continuous on a set C such that $P(X \in C) = 1$.

- If $X_n \rightsquigarrow X$, then $g(X_n) \rightsquigarrow g(X)$.
- If $X_n \xrightarrow{P} X$, then $g(X_n) \xrightarrow{P} g(X)$.
- If $X_n \xrightarrow{\text{as}} X$, then $g(X_n) \xrightarrow{\text{as}} g(X)$.

Prohorov's theorem

Recall from topology (consequence of Heine-Borel):

- A converging sequence in $\mathbb{R}^k$ is bounded.
- A bounded sequence in $\mathbb{R}^k$ contains a converging subsequence.

Prohorov's theorem is the stochastic version of these results.

Stochastic version of boundedness:

Definition. (Uniform tightness)
A collection of random vectors $\{X_\alpha : \alpha \in A\}$ is called uniformly tight, or bounded in probability if for all $\varepsilon > 0$, there exists an $M > 0$ such that

$$\sup_{\alpha \in A} P(\|X_\alpha\| > M) < \varepsilon.$$

Prohorov's theorem

Theorem. (Prohorov)

Let X_n be random vectors in $\mathbb{R}^k$.

- If $X_n \rightsquigarrow X$ for some X, then $\{X_n : n \in \mathbb{N}\}$ is uniformly tight.
- If $\{X_n : n \in \mathbb{N}\}$ is uniformly tight, then there is a subsequence X_{n_j} such that $X_{n_j} \rightsquigarrow X$ for some X.

Relations between stochastic modes of convergence

Theorem.
Let X, X_n and Y_n be random vectors.

(i) If $X_n \stackrel{\text{as}}{\to} X$, then $X_n \stackrel{P}{\to} X$.

(ii) If $X_n \stackrel{P}{\to} X$, then $X_n \rightsquigarrow X$.

(iii) For a deterministic c, $X_n \stackrel{P}{\to} c$ if and only if $X_n \rightsquigarrow c$.

(iv) If $X_n \rightsquigarrow X$ and $\|X_n - Y_n\| \stackrel{P}{\to} 0$, then $Y_n \rightsquigarrow X$.

(v) If $X_n \rightsquigarrow X$ and $Y_n \stackrel{P}{\to} c$ for a deterministic c, then $(X_n, Y_n) \rightsquigarrow (X, c)$.

(vi) If $X_n \stackrel{P}{\to} X$ and $Y_n \stackrel{P}{\to} Y$, then $(X_n, Y_n) \stackrel{P}{\to} (X, Y)$.

Useful consequence: Slutsky

Lemma. (Slutsky's lemma)

Let X, X_n and Y_n be random variables. Suppose $X_n \rightsquigarrow X$ and $Y_n \rightsquigarrow c$, for deterministic c. Then

(i) $X_n + Y_n \rightsquigarrow X + c$,

(ii) $Y_n X_n \rightsquigarrow cX$,

(iii) $X_n / Y_n \rightsquigarrow X/c$ if $c \neq 0$.

Proof.

Theorem 1.11 (v) + continuous mapping. □

→ ex 1.13